Geometry in Common Schools

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AN

ESSAY

UPON THE

STUDY OF GEOMETRY

IN

COMMON SCHOOLS.

BY

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PREPARED AS AN INTRODUCTION TO "FIRST LESSONS IN GEOMETRY, BY A. CROSBY, ETC."

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INTRODUCTION.

The importance of an earlier and more general study of Geometry, the expediency of bringing it into our common schools, and the need of a text-book adapted to that purpose, were, more than seven years ago, subjects of frequent conversation between myself and the author of the following treatise. I was strongly urged by him to undertake the preparation of a book. But no satisfactory plan occurring to me, the subject was dropped. Recently, however, Professor Crosby has again turned his attention to the subject, and, as no work had appeared which met our views, determined himself to make an attempt to supply the want which we had so long felt. The treatise now published is the result of this determination.

I shall avail myself of the opportunity offered by the publication of this work, and of the space afforded me by the author, to suggest some reasons for the introduction of Geometry into common schools, and certain principles, which should direct in the preparation of a text-book for this purpose.

The study of Geometry is admirably *adapted to the powers of the youthful mind.* Its ideas are *elementary.* The ideas of form and size are among the earliest which children acquire. Why should we not continue to direct their attention to ideas so early awakened? Why should we not extend their knowledge of subjects, towards which the mind so

naturally turns, and, while thus encouraging the mind to activity by leading it in the very path which itself has chosen, secure, instead of confused notions, distinct and accurate ideas?

This cannot be difficult. A child can distinguish between a straight line and a curve; can understand the nature of an angle, a triangle, a parallelogram, a square, or a circle; can compare magnitudes, and apprehend the relations of equality and inequality.

In Geometry, moreover, if anywhere, *one thing can be learned at a time*, a principle of admitted importance in the education of the young, though practically far too little regarded.

Again, Geometry is, in many respects, *more elementary and less difficult than Arithmetic.* It is true, that Arithmetic is studied in the child's school; Geometry, in the college. Arithmetic is universally regarded, and spoken of, as an easy study; Geometry is, by some at least, regarded as sufficiently difficult. But the difficulties of Geometry are present or recent difficulties; those of Arithmetic have been forgotten with the griefs of childhood. Arithmetic has been a subject of study from infancy. One of the earliest mental operations which the child learns is to count. Yet it is an operation of no little difficulty. The relations of different numbers are to be learned, and a name to be remembered for each particular number; in other words, for each combination of units. The *number* of the things counted is to be *abstracted* from the *things* themselves. Is not this at least as difficult as to distinguish varieties of form, or to learn the relations of different magnitudes? Is it not as easy to apprehend the distinction between a polygon with three angles, and one with four, as between the abstract numbers 3 and 4? Is it not as easy to see that two equal straight lines will coincide, as to see that 9 times 9 are 81? Or, to understand the definition of a straight line, as to appreciate the local value of figures? Is not, in short, the idea of extension as elementary as that of number? Will a child amuse himself by cutting a sheet of paper into some definite number of pieces, or by endeavouring

to produce some particular shape? by counting the pieces, or by comparing the forms with each other, and with other known forms?

"But why," it will be asked, "is the study of an *elementary* subject frequently so difficult to young men of more mature minds?" Partly from the very fact of the maturity of their minds, or, rather, of their mental habits. They have long thought and spoken loosely of the distinctions of form and magnitude, and have never given a moment's attention to the careful and discriminating consideration of such subjects. Is it strange, that, when young men with such habits undertake to acquire, in a short time, perfectly correct notions of so many objects of study, either entirely new, or seen in new relations, they should find difficulty? Here are serious obstacles to be encountered, aside from any intrinsic difficulty in the subject. First, the subject is new, and every thing connected with it strange. Then, at the age at which they usually commence the study, they cannot, as they might at an earlier period, afford time to stop on each principle as it occurs, and revolve it, and view it on all sides, till they become thoroughly acquainted with it, before looking at another. Principle after principle must be learned in rapid succession; and, if a single principle is passed before it is perfectly known, the mind instantly becomes confused, and the study difficult.

Another obstacle is found in the very *simplicity* of the subject. The topics first considered are so simple, that young men frequently cannot be persuaded to dwell upon them, and give them thought enough to make them perfectly familiar. Thinking it beneath them to sit down to learn the definition of a straight line or an angle, they give but little attention to the first principles, and wait for something more worthy of their study, till they find themselves lost in difficulties resulting from insufficient acquaintance with those very principles which they deemed so insignificant.

Now, at the age when we would have this study commenced, there is ample time to make every term and principle, as it occurs, perfectly familiar, and to illustrate and im-

press it by all the explanation which the subject admits. The pupil, moreover, at this age, will not take offence at the simplicity of the first principles, but will study them the more zealously on account of that simplicity, which brings them within his reach.

Another consideration should be regarded in estimating the difficulties experienced by more advanced students. Suppose young men should enter college, unacquainted with the first principles of Arithmetic, would they find no difficulty in that subject? Or suppose them utterly ignorant of Orthography, and that their college recitations consisted of spelling-lessons to be committed to memory, and accurately delivered in the recitation-room. Should we hear no complaints of the difficulty, the *impossibility*, of learning the lessons? And might it not then be argued with equal force, that acquisitions so difficult for college students certainly ought not to be required of the youth of an academy, and far less of the children of a common school?

Geometry will *interest the young* no less than the more mature. The mind delights to be fully employed upon subjects not beyond its reach. Geometry condescends to the powers of the young, while, at the same time, it furnishes abundant employment to the most mature. Subjects connected with it are among the first that interest the young mind, and the interest need not flag, so long as new truths remain to be considered; in other words, till the boundless resources of Geometry are exhausted.

The celebrated and excellent Pascal was in his early youth purposely restrained by his father from the study of Geometry, lest he should become so much interested in it as to neglect other studies. But the boy could not be prevented from casually hearing the conversations of the mathematicians who frequented his father's house. He heard and was interested; and, when he was about twelve years old, his father found him one day, in his room, alone and busy with a geometrical diagram. He had demonstrated, unaided by any book or teacher, the proposition, that the sum of the angles of a plane triangle is equal to two right angles. The father, notwith-

standing his previous caution, was pleased with the result, removed the interdict upon Geometry, gave him a Euclid, and encouraged him to study it.

A mathematician is said to have hired an individual to study Geometry, paying him wages, as for any other labor. At length, however, he told his pupil, that he could no longer bear the expense, and must forego the service. The pupil replied, that, if the mathematician would continue to teach him, he would willingly study without pay. This arrangement was made, and the study was zealously pursued, until at last the teacher informed his pupil, that he could no longer afford the time to teach him, but must employ it in earning something for his own support. The pupil replied, "If you must earn money, why not earn it as well by teaching me Geometry as in any other way? I will most gladly, not only study without pay, but pay you liberally for teaching me." The pupil was interested, and the study had become a source of enjoyment to him.

Such, substantially, I believe, will always be the result of thorough and well directed study of Geometry. One can hardly entertain clear and exact ideas of the properties and relations of magnitudes without a feeling of delight. And not only is the contemplation of the objects themselves, and their relations, a source of pleasure, but equally so is the consciousness of perfectly clear and accurate knowledge in respect to them, — knowledge free from every shadow of doubt, either as to its truth or its precision.

Again, no study is better adapted than Geometry to *discipline the minds of the young*. It is within their grasp; it interests, excites, tasks, and stores the mind; — not only stores it with useful knowledge, but furnishes it with *valuable habits*. This, which should be the grand object of intellectual discipline, — the formation of good mental habits, — is far too little regarded in our schools. The great effort, too often, is merely to communicate what is called *practical* or *useful knowledge*.

The storing of the mind with facts and principles for future use is indeed important, but it is still more important to

secure habits of right mental activity; habits of accurate perception, of cautious and exact reasoning, of love for truth, of modest self-reliance, of untiring application, and of profound and continued attention. That all these habits are cultivated by the study of Geometry will not be denied. "Pure mathematics," saith Lord Bacon, "do remedy and cure many defects in the wit and faculties intellectual; for, if the wit be dull, they sharpen it; if too wandering, they fix it; if too inherent in the sense, they abstract it." *

The effect of such mental habits, or of the want of them, will be felt in the *studies of the whole course of education*. The principles of Arithmetic should be demonstrated as rigorously as the propositions of Geometry. Unless this is done, Arithmetic is not *learned*. How much better this will be done by one accustomed to geometrical demonstration, needs scarcely a remark.

It is not out of place to remark here, that, in comparing the difficulties of Arithmetic and Geometry, the latter should be compared, not with Arithmetic, learned, as it too often is, by rote, with reference merely to mechanical practice, but with Arithmetic studied as it should be, intelligently and demonstratively. Arithmetic, thus studied as a science, will be found, in general, not less difficult than Geometry, and will, in many cases, more severely task the youthful mind.

The memory, it must be remembered, is not the only faculty to be cultivated. Yet this is, too often, the faculty chiefly developed in our schools, even in teaching Arithmetic. The proper teaching of Geometry will correct this error. A new method of study will be required, other faculties developed, and a change of mental habits effected, which will be most beneficially felt in all other studies, as well as in the whole subsequent life.

The *practical utility* of Geometry is too obvious to need discussion.

Some, however, while they acknowledge the practical utility of Geometry, and its appropriateness and value as a means

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* Advancement of Learning. Book II.

of intellectual discipline, will perhaps object, that *the common school is not the proper place for it.*

To this I reply, that it is the very place, and that, too, for several reasons. Can good habits be formed too early? Shall we gain any thing by delaying till young men come into the higher schools and the colleges? till bad habits are fully formed, and confirmed by long practice?

Again, multitudes who never enter the halls of a college, many who never enjoy the instructions even of an academy, whose whole school education is completed in the unpretending way-side school-house, need the knowledge of Geometry in the business of every day of their lives. Carpenters, wheelwrights, workers in tin and in the other metals, mill-wrights, land-surveyors, measurers, engineers, designers, navigators, cannot all be expected to have the advantages of collegiate or of academic education. Must they therefore be deprived of this so valuable discipline, and rest satisfied with the merely mechanical application of geometrical truths, without that knowledge of principles which would contribute so greatly to their own ease and satisfaction, and to the security of the interests intrusted to them?

And not only those who will need to apply the results of geometrical investigation, but *all* who receive their education in the common schools, should have the benefit of this discipline. This remark applies as truly to females as to males. For their intellectual strength, and consequent influence and respectability in society, they need this invigorating discipline. And we can scarce estimate too highly the advantage in educating the next generation, if mothers generally had the benefit of such training, so as to excite their children, by the true intellectual stimulus of sympathy, to the same habits of exact thinking and reasoning.

Still, it may be said by some (few, however, I hope), that it is better not to introduce new studies into the common school, but to confine it strictly to its *proper sphere.* But what is its proper sphere? In very many schools, till within a few years, the almost exclusive objects of attention were Spelling, Reading, Writing, and Arithmetic. English Gram-

mar and Geography, since become so common, were, in many places, unknown. At the same time, the knowledge acquired, even of Arithmetic, was very limited; many, especially females, aiming at nothing beyond the four Simple Rules and perhaps Reduction.

Many then feared, and some still fear, that the introduction of other studies would occasion the neglect of Spelling, and, in the words of a worthy patron of that system, of "the three R's, Reading, Riting, and Rethmetic." But what has been the fact? At the present time, Geography and English Grammar are common studies in all our schools; Arithmetic is universally studied, and, by most, to a far greater extent than formerly. At the same time, many new studies have been introduced, and among them Algebra, a subject certainly not more elementary than Geometry. But do the youth of the present day spell less correctly, and read less fluently and intelligently, than the youth of our schools thirty or fifty years ago? Far otherwise, we believe. In fact, a great part of the time then spent in school was wasted, for want of something to awaken interest, — *something to do.* Almost the whole time of the school was occupied in reading and spelling, and that in no very intelligent way, as is testified by the unnatural, monotonous voice, and the absence of all conversational tones, so often observed. Spelling, then, we think, — reading, we know, — is better taught now than formerly; and all the additional studies are clear additional gain.

Why then stop where we are? Our schools are longer than they then were, the children more at leisure to attend them. Will it not be still an additional and important gain, if we can secure in them the commencement of another science of so great utility as Geometry?

It is interesting to observe the progress made in the knowledge of Geometry, and in its diffusion through various classes of the community, during a period of twenty-five hundred years. We first hear of it, as an occult and mysterious science, among the Egyptian priests. We next find it in Greece, among the philosophers and learned men; — cultivated by Thales, Pythagoras, Plato, and their schools, and by Euclid,

Archimedes, and Apollonius. Thales is said to have brought it into Greece, about six hundred years before the Christian era; he is also said, which indicates the state of the science at that time, to have himself discovered that the angle inscribed in a semicircle is a right angle, and to have testified his joy by a sacrifice to the Muses! Pythagoras and Plato also are said to have extended the bounds of geometrical knowledge; the former, who lived about 550 B. C., being understood to have discovered that elegant proposition, which still bears his name, respecting the squares described on the sides of a right-angled triangle. He also is said to have expressed his joy and gratitude to the gods, by the sacrifice of a hundred oxen. Whatever may have been the facts in regard to these discoveries and sacrifices, the manner of their mention sufficiently indicates the limited extent of the science at that time, even among the greatest philosophers. Euclid, who lived about 280 B. C., has left us more abundant evidence of the state of the science in his time, in his "Elements of Geometry," a work which held its ground as the principal, almost the only, text-book on the subject for more than two thousand years; until his name became a synonyme for the science, and men spoke of studying, not Geometry, but Euclid. Later still, Archimedes and Apollonius distinguished themselves in the higher departments of mathematical science; Apollonius, particularly, by a most valuable treatise on the Conic Sections. Archimedes, "the most profound and inventive genius of antiquity," is celebrated, not only for his mathematical science, but for his mechanical skill, by which he defended Syracuse, for a considerable time, against the utmost exertions of a Roman army, and for the boast, that, if he had a place on which to fix his lever, he would move the world.*

Now, during all this time, and for many centuries after, the knowledge of Geometry was confined to the philosophers, — to the few. Out of the schools of philosophy, among the mass of the community, such science was utterly unknown. The universal diffusion of knowledge, as of all other blessings,

* Δὸς πoῦ στῶ, καὶ τὸν κόσμον κινήσω.

is the suggestion of Christianity. The properties of the circle and the triangle, at whose discovery Thales and Pythagoras are said to have been so elated, are now known to the tyro. Pascal, at the age of sixteen, composed a treatise on the Conic Sections, in which he gave, in a single proposition and four hundred corollaries, all that had come down from Apollonius, "the Great Geometer" of antiquity. And what one boy of sixteen may write, another boy of sixteen may learn. We believe that Geometry, instead of being confined, as formerly, to philosophers, or, as more recently, to an educated class, will, at a day not far distant, be introduced into the common schools all over our country, and brought within the reach of every boy and girl in the community.

This is the *proper sphere* of the common school; not to communicate a fixed amount of knowledge, the same to our children as to our fathers, but to communicate continually additional knowledge, and to produce higher and higher degrees of intelligence; — when men of science extend the bounds of knowledge, to diffuse that knowledge, till the world enjoys its benefits. This *universal and progressive diffusion of knowledge* constitutes the *proper sphere* of the common school.

But if Geometry is to be studied in our common schools, in what shape shall it be presented? What principles should direct in the preparation of a text-book for this purpose?

1. The *definitions* should be *perfectly clear* and *exact*. A definition should equally avoid excess and defect. It should express neither too much nor too little. It should be so full as perfectly to identify the object defined, but should not include properties the possibility of whose combination is yet to be proved.

Thus, Legendre's definition of the circumference of a circle as "a curved line, all the points of which are equally distant from an interior point, called the centre," * does not

* "La *circonference du cercle* est une ligne courbe, dont tous les points sont également distants d'un point intérieur qu'on appelle *centre*.

identify the circumference of a circle; but is equally applicable to any line whatever drawn upon the surface of a sphere. It should be defined, "a curved line *in a plane*, all the points, &c."

Again, when, before any demonstration of the properties of quadrilaterals, the square is defined as "a quadrilateral, which has all its sides equal, and all its angles right angles"; the rectangle, as one "which has its opposite sides equal, and its angles right angles"; the parallelogram, as one "which has its opposite sides and angles equal," or "its opposite sides equal and parallel," — these are examples of excess in definition. How do we yet know that a quadrilateral *can* have, at the same time, its opposite sides equal and parallel, or its opposite sides and angles equal, or its opposite sides equal and its angles right angles, or all its sides equal, and its angles right?

Clearness and simplicity in the definitions are promoted by a natural *order of succession*. Of the quadrilaterals, for example, the square is, in many of the books, defined before the rectangle, and the rectangle before the parallelogram, that is, the *species* before its *genus*. Whereas the true order of science requires us to proceed from the more to the less general, adding, at each step, only the necessary limitations or specifications.

An object should be defined by means of that *distinguishing property*, from which its other properties may be most easily and satisfactorily deduced. The fact that *parallels* never meet seems to be less their distinguishing property, than a consequence of some other property. This property furnishes no convenient means of drawing parallels, nor any practical test of parallelism. It is much more satisfactory and convenient to define them as *having the same direction*.

2. The *propositions* should be *enunciated with the utmost precision*. A defect in this respect sometimes amounts to a gross error. Thus, in some of the books, we find this prop-

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" *Le cercle* est l'espace terminé par cette ligne courbe." — *Éléments de Géométrie*, 11me Éd., p. 33.

osition. "If the product of two quantities be equal to the product of two other quantities, two of them" (any two, of course) "may be made the extremes, and the other two the means, of a proportion"; e. g. $4 \times 8 = 2 \times 16$; then, making 4 and 2 the extremes, we have $4 : 8 = 16 : 2$; or, as the product of the extremes is equal to that of the means, $4 \times 2 = 8 \times 16$, or $8 = 128$. It should be, the *two factors of one product* may be made the extremes, &c. Nor is this a mere captious objection. I have repeatedly known students to make the mistake, and find it out only by trying to verify the result.

Again, a good enunciation distinguishes, and marks the distinction with great care, between *hypothesis* and *conclusion.*

3. The *most rigorous exactness of demonstration* must be preserved. We want no tentative or experimental methods of proof. Empiricism is as bad in mathematics as in medicine. We want no practical results to be learned by rote, without proof. We must have proof, — infallible proof, — demonstration. The reasoning may be simplified, and reduced to the comprehension of the young, by multiplying and shortening the steps, if need be; but still *it must be demonstration.*

4. The memory is aided, fresh interest awakened, and the whole mind invigorated, by the *generalization of geometrical truths;* a process which connects under one enunciation several apparently distinct propositions, and shows them to be only particular applications of a more general principle, — specific forms of a generic truth.

5. A book designed for elementary schools should abound in *minute and familiar illustration*, both of terms and principles. The definitions and propositions should be expressed as concisely as possible, so that they may be easily remembered, and conveniently quoted. But the terms used should be explained with great care, and such remarks added as will connect and show the relation between the rigorous expressions of Geometry, and the looser language of ordinary conversation.

The practical application of principles should be set forth, not to strengthen the proof of a proposition, — for the infallible nature of demonstration, and the impossibility of increasing its certainty by additional evidence should be constantly insisted on, — but rather to illustrate the meaning of the abstract principle, to show how readily it applies itself to practical results, and connects itself with common things, and so, at the same time, to aid, to interest, and to benefit the pupil.

It may be thought, that the teacher should supply the necessary illustrations. He should, indeed, so far as his time permits; but yet, for various reasons, besides the want of time, the burden should not be wholly thrown upon him.

In the first place, many will be called upon to teach Geometry who are not particularly interested in it; some, perhaps, who are not very familiar with it; and some, possibly, like a teacher I once knew, who thought that "Euclid and English Grammar could be learned only by committing them to memory." To such teachers, illustrations, unless suggested by the book, will not be likely to occur.

Another difficulty will be experienced by the most accomplished teachers. Explanations, if first suggested during the recitation, will not generally be appreciated or remembered. The pupil should *study them with his lesson;* he should reflect upon them, and see for himself their connection with the subject. He will then be prepared to appreciate any additional remarks from his teacher, and to ask intelligent questions of his own. The more abundantly illustrations are furnished by the book, the more readily will additional illustrations occur to both teacher and pupils.

It may not be out of place here to remark, that a difficulty, of far greater magnitude, indeed, but of the same nature, is occasioned by *faulty and inelegant definitions and propositions.* If the fault be pointed out before the lesson have been learned, the subject is strange, and the correction not understood or remembered; if afterwards, the faulty expressions will then have been fixed in the mind, and cannot easily be eradicated.

" But after all this simplification, illustration, and improve-

ment of the form and arrangement of the definitions and propositions, 'there is no royal road to Geometry.'" True, the subject cannot be mastered without labor, but that labor may be intelligent and well directed. The height must be scaled, but it need not be approached on its most precipitous side. The road must pass over the loftiest summit, but, by beginning early, the ascent may be gradual and easy.

S. C.

Dartmouth College, Jan. 22, 1847.